God, Infinity, and Language

Semiotic Perspectives on the Philosophy of Mathematics

by

Noel Arteche

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Professor: Jan Baetens
Student number: r0773753

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Abstract

This essay is a semiotic study of mathematical language with a view to getting some insights into central issues of mathematical philosophy. In the first half, Brian Rotman's theory of corporeal semiotics is reviewed and explained at length, a model considered one of the most ambitious projects in postmodern epistemology and philosophy of science. In the second part, a concrete mathematical sign ($1+2+3+\cdots = -\frac{1}{12}$) is studied using different semiotic theories, including Rotman's.

Contents

1 Introduction

1.1 God, infinity, and language

German mathematician Georg Cantor (1845-1918) once said: "I entertain no doubts as to the truths of the transfinites, which I recognized with God's help". Something similar is what he thought about the well-ordering theorem, an obvious "law of thought" that had to be true in the eyes of God. Cantor, whose work on transfinite numbers brought back the problem of infinity to mathematics and who standardized contemporary mathematical language through the use of set-theoretical terms is the best example of how God, infinity, and language are at the core of the philosophical discussion in mathematics.

Unfortunately, due to the image of mathematics as the vehicular language for technoscience under capitalism, we never think about those three issues at length. Mathematics is only thought of as the language of nature in a very restricted referential dimension. We tend to think of mathematical language as pointing to some preexisting Planotic ideas, objective, eternal and conveying unquestionable truth. Ideas, undoubtedly, very appropriate for the "objective" natural sciences, on whose allegedly rock-solid foundations technoscience relies.

Due to this narrow idea of what mathematics is and can be, the efforts to better understand them from a humanistic perspective have been few and dominated solely by formal linguistics and analytic philosophy. It is clear, however, that given the new perspectives on language developed ever since the mid 20th century, let that be from cultural studies to postmodern thinking, from literary theory to linguistics, new approaches should be considered when thinking about mathematics and its role as language and culture.

This essay focuses precisely on how semiotics can shed light on these issues. Despite semiotics being somewhat popular in mathematical education and literacy research, its relevance when it comes to the actual philosophy of mathematics is rather limited. Charles S. Pierce, a mathematician himself, never fully tackled the question and no real efforts have been directed in this area until recently.

The very first person to approach philosophy of mathematics from semiotics has been Brian Rotman, who has been working on it since the late 1980s. Rotman has developed a semiotic model that we may call the "corporeal semiotics for mathematics", first sketched in his book *Signifying Nothing: The Semiotics of Zero* (1987) and later extended in his compilation of essays *Mathematics as Sign: Writing, Imagining, Counting* (2000). Undoubtedly, though, his most influential work is his book *Ad Infinitum: The Ghost in Turing's Machine* (1993), where he addresses the problem of infinity and opens the door to the so-called non-Euclidean arithmetic. Inspired by Rotman, other authors have worked on their semiotic approaches to the philosophy of mathematics, among which we might refer to those of Mortensen and Roberts (1997).

This paper has two distinct parts. In the first half, I review Rotman's model of corporeal semiotics as devised in *Towards a Semiotics of Mathematics* (1988) and *Ad Infinitum* (1993), whose influence comes basically from Saussure and, more importantly, Pierce. I emphasize his approach to mathematics as language and cultural activity and his insights into the problem of infinity. Finally, I explain some of the ideas and logical foundations for non-Euclidean arithmetic and its rejection of Platonism.

The second part of the essay is the semiotic analysis of a sign, the infinite summation of natural numbers being equal to $-\frac{1}{12}$. I apply three classic semiotic models to it as well as Rotman's ideas, to show how infinity is thought through signs and to give some notes on the meaning of equality as the embodiment of sign prediction. Therefore, the final goal is, in short, to see how semiotics can offer an alternative approach to the philosophy of mathematics and provide new insights.

1.2 Philosophical preliminaries: the problem of infinity through Platonism, formalism and intuitionism

I assume basic familiarity with semiotics, though the relevant concepts will be reviewed when necessary. On the other hand, we shall briefly outline the central issues of mathematical philosophy that appear throughout this text.

The *problem of infinity* is the seemingly irreconcilable opposition between *potential* and *actual* infinity. Potential infinity is the notion that infinity is something that exists only *in potentia*, as something which we can approach but never reach. A variable x that "tends to infinity" conveys a potential view of infinity, as it gets bigger and bigger but it never really *is* infinite. In the same way, one might think of the succession of natural numbers, $0, 1, 2, 3, \ldots$ as a potential infinity, where a new element can always be generated.

On the other hand, the actual infinity refers to the possibility of having a finished and given infinite object, such as the set of natural numbers, $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ where all numbers exist at any given time, finished, infinite and at the same time perfectly closed and bounded inside a definite mathematical object like a set. The actual infinity was a central notion of Georg Cantor's theory of sets and transfinite numbers, which threatened the foundations of mathematics at the turn of the 20th century. Two philosophical movements emerged as possible solutions.

On the one hand, David Hilbert's (1862-1943) program, known as *formalism*, tried to show the consistency of the mathematical apparatus through solely finitary means. The name formalism stands for the insistence in separating the mathematical activity into a formal level, where mathematics is considered as mindless manipulation of meaningless marks on paper, and a metamathematical level, where the “substantive thinking” about mathematics takes place.

Hilbert’s formalism tried to convince L. E. J. Brouwer (1881-1966), whose *intuitionism* tried to build mathematics without using actual infinities. This meant a revisionistic approach based on the idea the mathematics is essentially an activity of the mind, where reason has access to basic intuitions about mathematical objects, independent of their representation in writing. Besides, Brouwer defended intuitionistic logic, where the principle of excluded middle no longer holds and things must be constructed to show their existence.

Despite their undeniable influence, neither of them managed to be a satisfactory philosophical model. As a result, most contemporary mathematicians resort to *Platonism*, according to which mathematical objects exist in a preexisting realm of ideas as perfect, eternal objects, conveying universal truth, and mathematical language and its signs are nothing else than a reference to these Platonic objects. Under Platonism, both actual and potential infinities coexist.

2 Rotman’s corporeal semiotics for mathematics

2.1 The model: Person, Subject, and Agent

Brian Rotman motivates the search for a semiotic framework for mathematics as an attempt to answer two questions. First, “what is the nature of the mathematical subject?”And, second, “could a semiotics of mathematics that responded the first question add something new to the stale triad of philosophies [Platonism, formalism, and intuitionism] that dominate the field of mathematical philosophy?” (Rotman 2000, p. 42)

His inquiry starts with a preliminary approach to the question “what is mathematics?”. Clearly, it is an intellectual activity, an activity of thought. But, as Rotman phrases it, “*being thought* in mathematics always comes woven into and inseparable from *being written*” (Rotman 1993, p. X). Thus mathematics is an intellectual activity done through writing. But writing is an activity that takes place in a certain social, cultural and historical context, which means mathematics is also a cultural activity related to writing. And, in being closely related to writing, it is closely related to language. Mathematical language, however, is a very particular sort of language.

If we analyse mathematical texts we will find a recurrent pattern: they combine a certain amount of natural language with a fair amount of very specific ideograms and symbols, manipulated according to specific rules. This is what Rotman’s calls the *Code*.

The Code consists of a careful combination of natural and symbolic language combined in a delicate tangle of statements and proofs in which we find *indicative* language ("let ϵ be a real number", "consider a vector space V", "there are infinitely many prime numbers", "problem P is NP-complete", ...) woven into *symbolic expressions* ($\epsilon < \delta$, $\Phi \vdash \Gamma$, $A = \{x : x \notin A\}$, ...) defining mathematical objects over which actions are projected in an *imperative* language: "*compute* the square root of ϵ", "*find* a basis for V", "*integrate* function f", "*reduce* problem P", etc.

The Code combines these three types of language (indicative, symbolic and imperative) in an sequence of pairs, each containing a theorem (an indicative statement) and a proof. But what exactly is a *proof*?

A proof involves the idea of an argument: a narrative that must convince us of the truth and logical necessity of the stated theorem. In fact, Charles S. Peirce's semiosis, the process by which an alarming sign is unfolded into a potentially infinite chain of signs to arrive at a satisfactory interpretant is very much the nature of a mathematical proof which is, in essence, a "thought experiment".

The mathematical Code invites the reader to engage in a sort of self-reflective thought experiment: some subjective agency imagines themselves acting to build a narrative that can convince them about a certain thing being the case; a narrative that can convert alarming signs into satisfactory signs. In Pierce's own words:

> "It is a familiar experience to every human being to wish for something quite beyond his present means, and to follow that wish by the question, 'Should I wish for that thing just the same, if I had ample means to gratify it?'. To answer that question, he searches his heart, and in so doing makes what I term an abstractive observation. He makes in his imagination a sort of skeleton diagram, or outline sketch of himself, considers what modifications the hypothetical state of things would require to be made in that picture, and then examines it, that is, observes what he has imagined, to see whether the same ardent desire is there to be discerned. By such a process, which is at bottom very much like mathematical reasoning, we can reach conclusions as to what *would* be true of signs in all cases"
>
> (Rotman 2000, p. 13)

This paragraph by Pierce is the very essence of Rotman's semiotic model for mathematics. What Peirce refers to as the "self" is what Rotman calls the mathematical *Subject* (the one who imagines the thought experiment, who creates and follows the proof) while the "skeleton diagram of the self" is what Rotman calls the *Agent*. The Subject reacts to the inclusive indicative elements of the code ("consider", "let", "prove"), while the Agent blindly executes the actions imperatively proposed as steps of the proof ("add", "count", "integrate").

The model, however, is incomplete. Let us recall that Pierce points out that three different relations may happen between signs and objects: the *symbolic* (Saussure's basic relation between signifier and signified of most words), the *iconic* (coupling produced by the homology of subject and object) and the *indexical* (produced by the physical context of the sign use). It is the third one of these relations that we are interested in. When analysing the mathematical Code we observe that there is no indexicality. The Code has no context. In fact, the Subject is a-historical, a-cultural, a-social; "a timeless voice from no one and from nowhere" (Rotman 1993, p. 74) who, most notably, is devoid of indexicality. There's no deixis, there's no "I", there's no indexical coupling between sign and object.

But can mathematics exist without indexicality? We said that mathematics is a cultural activity and thus it must be properly contextualized. Thus, we shall ask: Where is the indexicality? Where is the context? Definitely not in the Code, but rather in the *meta-Code*.

The meta-Code is, in opposition to formal mathematics, the place where mathematics is approached informally. It is the place where the context is explained, where mathematicians ask for "the idea behind the proof", a vantage point where we appreciate that the Subject belonging to the Code is nothing else than a truncation of third entity: the *Person*, who operates with the meta-Code.

In short, Rotman's semiotic model for mathematics consists of three elements: the Person, the Subject and the Agent. The Person is the entity interacting with the meta-Code; the Subject is an a-cultural truncation of the Person imagined by the Code; the Agent is the subject's formal and mechanical proxy who enacts its thoughts. As Rotman explains in an analogy with dreams, the Subject is the dreamer, the Agent is the dreamer's actor and the Person is the dreamer awake, telling and interpreting the dream. Figure 1 shows a simplified diagram of the model.

Now, what is a mathematical sign? According to Rotman, mathematical signs or assertions are essentially *predictions about signs*. For instance, the sign $2 + 3 = 3 + 2$ conveys the prediction that concatenating $||$ and $|||$ will give the same sign as concatenating $|||$ with $||$, were we to perform the actual writing of those lines. That was a simple example, but in something like $150 + 347 = 347 + 150$ we would not write it down ourselves. Instead, we would make our inner mathematical Subject launch a thought experiment (a proof), in which a mechanical proxy would be invoked, the Agent, who would be in charge of performing this counting action. The Agent's actions convince the Subject that were they to execute those actions, they would get

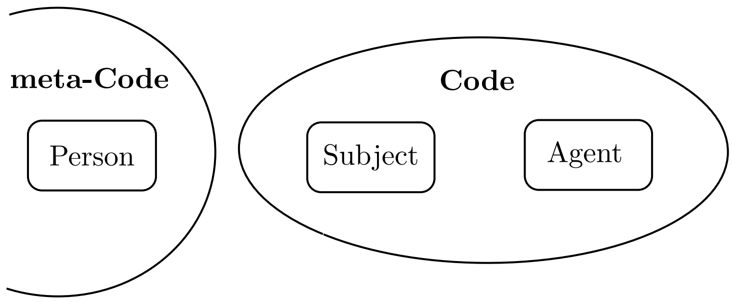

Figure 1: Rotman's Person-Subjet-Agent model.

the same results, thus complying with the prediction presented by the original sign/assertion.

Finally, we shall note how signs come into being. As opposed to Platonism, where mathematical signs only point to preexisting objects, Rotman thinks mathematical signs are cocreative, in the same way Saussure thinks that the relation between signifier and signified is conventional and not deriving from any sort of natural resemblance. Thus, after a careful study of the historical development of mathematics, one feels forced to acknowledge that signs shape mathematical ideas (signifieds) as much as mathematical ideas shape mathematical signs (signifiers).

2.2 Why *corporeal* semiotics?

At the beginning of this essay, I anticipated that Brian Rotman's semiotic project could be granted the adjective of *corporeal.* So far, however, the model is completely abstract and ethereal. We shall now pay attention to the nature of the Agent to understand why Rotman calls his semiotics "corporeal".

As an abstract automaton capable of carrying out any complex —or rather general[1]— mathematical computation, the role of the Agent could perfectly be played some sort of Turing machine: an abstract model of computation capable of carrying out mechanical processes with infinite capabilities[2]. Given the ideal nature of this Turing machine-like Agent, we would expect it to be able to perform infinite computations, especially those the Subject could never perform.

When we consider Rotman's model up to this point we are envisioning a semiotic framework for the existing mainstream mathematics. We are, in other words, giving a semiotic account of Platonism, where mathematical objects

[1]An abstract model of computation is said to have *general* power if it is equivalent to the universal Turing machine. Under the Church-Turing thesis, general calculability and general computability are considered to be the same and thus for Computability Theory anything that shall be computable is computable by a Turing machine.

[2]Turing grants his machine an infinite tape to write on and computability-theoretical notions of time and space are defined in terms of this ideal tape.

belong to a preexisting realm of ideas as perfect objects and we imagine our ideal machine making computations on them. This semiotic model accounts for a very specific —though mainstream and happily embraced by natural sciences— conception of mathematics. It answers Rotman's first question (what is the nature of the mathematical subject) but not the second one: could a semiotics of mathematics add something new to the field of mathematical philosophy?

He thinks so and suggests that this Agent shall not be infinite nor ideal. That, in fact, we can only project onto it the same scribbling capabilities that the Subject has in its finitary condition. In other words, the Agent shall not have an infinite tape to write on nor the ability to manipulate infinite objects. By doing this we are no longer capable of working with the actual infinity of natural numbers and we are forced to turn our backs on Platonism.

Given the fact that this new version of the Agent has finitary limitations, we cannot expect it, for instance, to infinitely keep counting. Were we to imagine a machine devoted to counting up to any number, we would meet the energy and entropy limitations of the universe. We would reach a point where the precision of counting would be inaccurate, where things would start to dissipate. Thus, for Rotman, we can only think of *corporeal* Agents, whose scribbling capabilities are similar to ours. Mathematics and its semiotics must be related to the physical act of writing and thus the name of "*corporeal* semiotics for mathematics".

2.3 Against Platonism: the rise of non-Euclidean arithmetic

In 1935, Soviet-era avant-gardist and absurdist poet Daniil Kharms (1905-1942) wrote *A Sonnet*. It starts as follows:

> An astonishing incident happened to me: I suddenly forgot what comes earlier, 7 or 8.
>
> I went over to my neighbors and asked them their thoughts on the matter.
>
> How great was their and my own astonishment, when they suddenly found they also couldn't remember the counting order. 1, 2, 3, 4, 5 and 6 they remember, but further on they forgot.
>
> We all went to the commercial store called "Grocery", which is on the corner of Znamenskaia and Basseinaia streets, and asked the cashier about our bewilderment. The cashier smiled sadly, took a tiny hammer out of her mouth and, after moving her nose a little, said: "I think 7 comes after 8 in the case when 8 comes after 7."
>
> (Ostashevsky 2013, pp. 34–35)

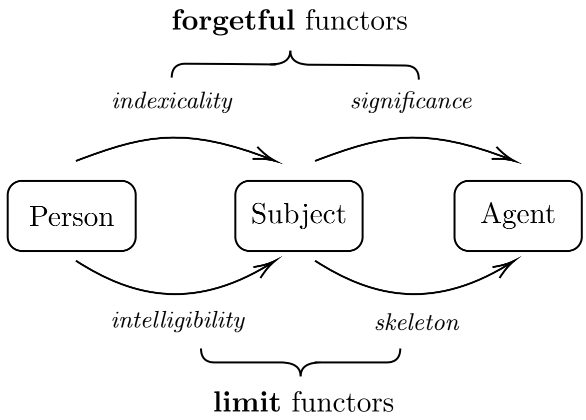

Figure 2: Forgetful functors and limit functors on Rotman's model (as depicted in Rotman 1993, p. 92). Forgetful functors convey the idea of truncation and produce non-Euclidean numbers, while limit functors show the limits to mathematical significance in new operators that are not semiotizable.

As absurd as it might seem, Kharms, whose children's poetry "foregrounds the —so to speak— subjective aspect of mathematics, in that it looks at mathematics as a *particular human activity*, something that people *do*" (Ostashevsky 2013, p. 33), was perhaps anticipating the most astonishing consequence of Brian Rotman's semiotic model: that counting may not behave as we think it does. That, perhaps, as Kharms put it, "numbers are not bound by order".

When embracing Rotman's Agent in its finitary form and condition and rejecting Platonism, we are forced to leave aside the notion of actual infinities and thus the finished set of natural numbers. Consequently, the notion that emerges from Rotman's analysis is that of *non-Euclidean numbers*.

When discussing the Person-Subject-Agent model we said that the Subject is a truncation of the Person and the Agent is a truncation of the Subject. This is expressed by Rotman in what he calls *forgetful functors*, conveying the idea of truncation. When going from Person to Agent *indexicality* is forgotten, while when going from Subject to Agent *significance* is forgotten.

Forgetful functors are reflected in a pair of *limit functors*, which express what elements of an entity are projected onto the next. The Person projects *intelligibility* onto the Subject, while the Subject projects its *skeleton* onto the Agent. Forgetful and limit functors are displayed in Figure 2.

The skeleton limit functor is what makes possible the thought experiments we have called proofs. However, this skeleton has a finitary limitation, which emerges in its most evident form when counting.

Rotman argues that when analysing this functor one reaches a conclusion similar to that of Kharms: that counting cannot extend infinitely and that

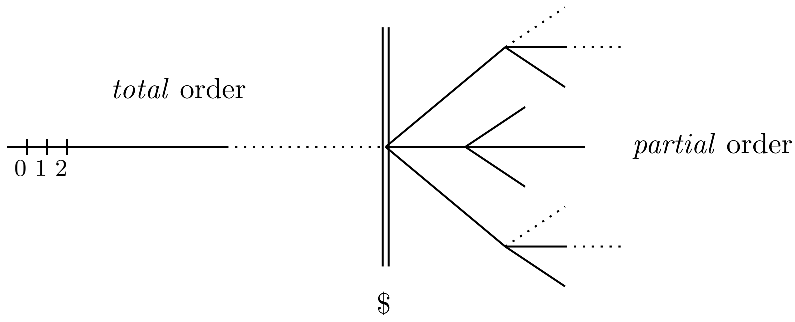

Figure 3: Total order holds until the \$-limit, where arithmetic is "locally Euclidean." (as depicted in Rotman 1993, p. 126).

the sequence of natural numbers should be replaced by a new sequence, what he represents by $1, 2, 3, \ldots \$$, where \$ stands for a *counting limit*, the point where the Agent's accuracy is no longer to be trusted and thus the total order of natural numbers is violated. As Kharms would say: numbers are (no longer) bound by order, and counting proceeds in a foggy partial ordering where arithmetic no longer behaves as we expect. This is depicted in Figure 3.

Some may be tempted to dismiss Rotman's non-Euclidean arithmetic as utter nonsense, but its logical motivation is rather solid and legitimate. In the same way that Lobachevsky's negation of Euclid's fifth axiom (the parallel axiom) opened the door to non-Euclidean geometry, Rotman advocates for a critical revision of the Peano axioms for arithmetic[3]. We can still maintain Axiom 1[4], but Axiom 2 may no longer hold: there may not be a successor function that works for every natural number. As a result, Axioms 3 and 4 no longer hold and, on the same grounds, Axiom 5, the foundation for the principle of natural induction, is no longer valid.

[3]The Peano axioms for the set $\mathbb{N}$ of natural numbers, first introduced by Giuseppe Peano (1858-1932) in his *Arithmetices principia, nova methodo exposita* (1889) and later modified by Bertrand Russell (1872-1970) are usually stated as follows:

1. 0 is a natural number.
2. There exists a *successor function* s such that for every natural number n there exists a *successor* of n which is also a natural number.
3. 0 is not the successor of any natural number.
4. If two natural numbers have the same successor, then they are the same number.
5. If a subset of $\mathbb{N}$ contains 0 and for every number it contains it also contains its successor, then it contains all the natural numbers.

[4]If we negate Axiom 1 then there cannot be any arithmetic. Under the von Neumann construction of natural numbers Axiom 1 is translated to "there exists the empty set $\emptyset$", which is to say: "nothingness exists". It does not make sense to put this postulate in doubt, at least if we want to do some mathematics.

Seen this way, Rotman's non-Euclidean arithmetic is as logically legitimate as Lobachevksy's non-Euclidan geometries, yet it is essentially finitary, to a certain extent reminiscent of Thoralf Skolem's project for the foundations of arithmetic at the beginning of the 20th century.

On the other hand, the intelligibility limit functor also announces a certain limit to mathematical knowledge. In the same way that in a film or a book there is only a certain amount of flashbacks that can be nested before the reader gets lost in the story, in mathematics new operators also reach a point where they are no longer semiotizable. Sum is repeated application of the successor function; multiplication is repeated sum; exponentiation is repeated multiplication; hyper exponentiation is repeated exponentiation... but it is already intricate to give meaning to it. Even though computers may be able to carry out more complex nested operations, this nesting may not be infinite, as it soon becomes impossible to semiotize as a consequence of the intelligibility functor. In other words, the Subject can never be more intelligent than the Person, and there are limits to the Person.

3 Semiotics in action: $1+2+3+\cdots=-\frac{1}{12}$

Now that we have covered at length Brian Rotman's framework for the semiotics of mathematics, we shall turn our attention to how semiotics can be used to understand mathematical signs and how it can offer new insights into the philosophical issues discussed at the beginning. Rotman considers his book *Ad Infinitum* (1993) an analysis of the sign '...' representing infinity, so we will study a sign where '...' appears. What follows is a case study of the summation of all natural numbers being equal to $-\frac{1}{12}$:

$$1+2+3+4+\cdots=-\frac{1}{12}$$

Of course, "this cannot be true!" Arguably, if we only add positive integers, we cannot get a negative fraction. In fact, the left-hand side of the equality corresponds to the series $\sum_{n=1}^{\infty} n$, which, indeed, is divergent in the usual sense of the term[5]. Nevertheless, this equality is present in many textbooks, it has important applications in physics and appears in popular science articles and educational videos. How is this possible?

There are different explanations for this apparent nonsense, but I will leave them for later. First, we will focus on the surprise element and apply Saussure's and Greimas's semiotic models, as well as Rotman's. Then we will

[5]A series $\sum a_n$ is said to be *convergent* when the limit $\lim_{n\to\infty}(a_1+a_2+\cdots+a_n)$ exists and is finite. A series that is not convergent is said to be *divergent*. Only convergent series are summable and can be given a value for their infinite sums. The infinite sum $1+2+3+\ldots$ is divergent and therefore not summable.

use Peirce's semiosis to see how mathematical reasoning unfolds to give an explanation.

3.1 Rotman

We shall start by applying Rotman's corporeal semiotics to this sign. As we said, Rotman thinks of mathematical signs or assertions as basically "predictions about signs". Were we to perform the infinite summation of all natural numbers, we would or would not get as a result $-\frac{1}{12}$. As we saw, the mathematical Subject reads the equality, which belongs to the Code, and invokes the Agent, a mechanical proxy in charge of performing the sum in such a way that convinces the Subject that if they performed it themselves, it would give the predicted result.

But can the Subjet think of a mechanical process for the Agent to execute that gives as a result the desired $-\frac{1}{12}$? Not really. This is the point where some of the shortcomings of Rotman's model become obvious. His Person-Subject-Agent theory explains how proofs are read and signs are interpreted, but he does not manage to give a convincing explanation of how these signs come into being; he never explains how proofs are created.

In a certain way, Rotman assumes proofs already exist and describes how they are read, how mathematicians make sense of them. One could feel a certain Platonic smell in this: proofs being preexisting finished objects that we read but never create. In fact, I would say that, despite their many limitations, either formalism or intuitionism give better explanations as to how proofs come into being.

We must remember, nevertheless, that Rotman's semiotics is corporeal. We can analyse the previous sum from the standpoint of the newly discovered non-Euclidean arithmetic. Then the sign would be rewritten with the limit symbol \$:

$$1 + 2 + 3 + 4 + \cdots + \$ + \cdots = -\frac{1}{12}$$

Now we may argue that perhaps it is because we are adding elements beyond the \$-limit that the result no longer makes sense. In fact, this aligns with Rotman's ideab of non-Euclidean arithmetic being "locally Euclidean" when the operands are less than the \$-limit, but being difficult to grasp if any of the operands happens to be beyond that turning point.

Though arguably creative, the current state of non-Euclidean arithmetic is too rudimentary to be able to offer further insights into our sign. It is time to take a step back and look at it from a broader semiotic perspective and apply other models to it.

3.2 Structural semiotics: Saussure and Greimas

Ferdinand de Saussure's (1857-1913) structural semiotics considers language to be a system of signs. Saussure's key ideas are three. First, every sign can be unfolded into a material and an ideal component, the *signifier* and the *signified.* Second, the link between signifier and signified in a language is conventional. Third, a sign's position inside the language is not due to any intrinsic properties of the sign but due to oppositions, to the differences with the rest of terms.

When looking at the sign $1+2+3+\cdots=-\frac{1}{12}$, our attention is first driven by the left-hand side of the equality, where after the concrete sum of three terms we find the symbol '$+\ldots$'. On the one hand, we find a signifier, the combination of the symbols '+' and '$\ldots$' (which have received their meaning in completely conventional and arbitrary ways), and a signified, which could be phrased as "repeated sum for the elements that may come". This "repeated sum" is a *process*: a sequence of instructions to be executed one after another to obtain some result. In particular, an *infinite* process, that could be continued forever.

What is this sign opposed to? We might first think that processes are opposed to *objects*, an idea that seems to convey the conflict between potential and actual infinities. But, in fact, there is no contradiction between processes and objects. Many mathematical ideas can be thought of both as a process and as an object. For instance, the function $f(x) = x^2$ can be thought of both as a process by which one inputs a value x, squares it and gets a new value, but also as an object: a static map or correspondence between every real number and its square. The same happens with an algorithm, which is both a process and, at the same time, an object, when imagined as a Turing machine. Thus the opposition is in the combination of infinity and process. In other words, the reason we feel alarmed and puzzled when reading $1+2+3+\cdots=-\frac{1}{12}$ is that the left-hand side represents an *infinite process* while the right-hand side stands for a *finite object.*

To go a step further we now turn to Algirdas Julien Greimas's (1917-1992) *semiotic square*, where meaning is no longer a binary opposition. A sign s_1 is opposed to a sign s_2 in a relation of *contraries.* Besides, we have the *contradictory* elements, obtained when negating s_1 and s_2. The *positive term*, s_1, will be "infinite process", while its contrary is "finite object"[6]. The negation of "infinite process" is, naturally, a "finite process", while a "finite object" is negated into an "infinite object".

As we can see in Figure 4, the *complex term*, obtained by the conjunction of

[6]Note that a semiotic square where we study a compound term (formed by two words like "infinite process" here) is a dangerous one, because we might not know if the negation should apply to the first or the second word in the term. However, as we saw, processes and objects may coexist in many mathematical concepts, so there is no confusion.

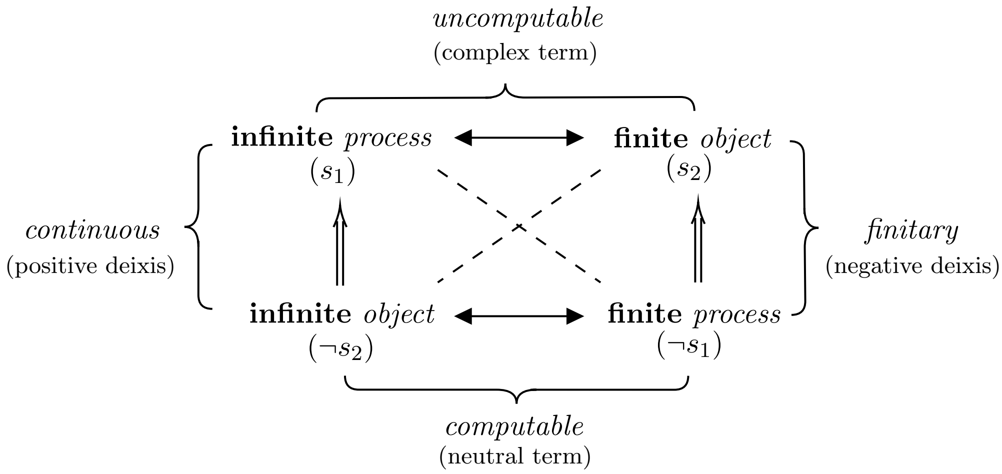

Figure 4: Semiotic square for *infinite process* vs. *finite process*.

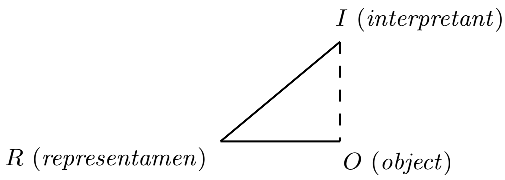

Figure 5: Schema of a Peircean sign.

"infinite process" and "finite object" is the realm of the *uncomputable*: infinite computations do not halt and thus the finite object is uncomputable. On the other hand, the neutral term is the realm of the *computable*. For instance, number π, which to the extent it is irrational is incommensurable (its decimal expansion is infinite) but which can be computed by a finite process to any degree of accuracy.

As a result, Greimas's semiotic square helps us understand why we are puzzled by the sign: it lies in the complex area and thus it is "uncomputable". Which, translated to the terminology of Rotman's semiotics, means we cannot imagine an Agent executing a computation that leads to the result $-\frac{1}{12}$.

3.3 Mathematical reasoning unfolding: Peirce's semiosis

Greimas's semiotic square helped us understand why we are puzzled by the sign, yet I said the equality $1+2+3+\cdots = -\frac{1}{12}$ does convey some mathematical truth. How is this possible?

Let us briefly discuss the mathematical reasoning for this puzzling equality in the form of the unfolding schema of a Peircean *semiosis*.

Charles S. Pierce's (1839-1914) dynamic conception of the sign, consisting of a ternary structure made by the *object*, the *sign*, and the *interpretant* (see

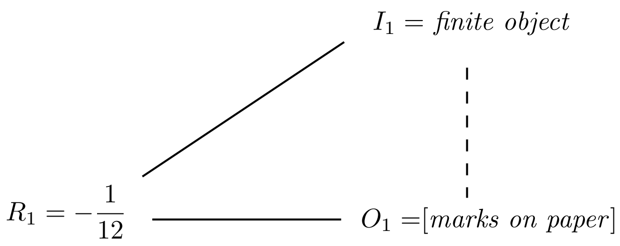

Figure 6: The right-hand side of the original sign represented as a Peircean sign. Since its interpretant is already satisfactory is does not produce further semiosis.

Figure 5), conceives semiosis as the process by which we go from object to interpretant, rendering this into a new sign that is susceptible of launching a further step of the semiosis. This chain of semiosis makes us closer to the "final interpretant", "the inevitable and universal discovery of truth" according to Peirce.

Pierce distinguishes between *satisfactory* signs (those where we are happy with the interpretant) and *alarming* signs (where we must launch a semiosis to find a temporary satisfactory interpretant). It is rather obvious that the sign $1 + 2 + 3 + \cdots = -\frac{1}{12}$ is alarming: we are not happy with its interpretant since it is contradictory. How do we transform the alarming sign into a more satisfactory sign?

Following Rotman, we say that a mathematical assertion is a prediction about signs, and thus we will need two separate semiosis: one for the left-hand side of the equality and another one for the right-hand side. The goal is to get to the same interpretant on both sides. If this is possible and the interpretants agree, then we will be able to make sense of the complete sign. If not, then the assertion will be considered false.

The right-hand side of the equality, however, is rather trivial: it is already a satisfactory sign. It is a perfectly valid finished object, a negative rational number. Its schematic representation as a Peircean sign can be seen in Figure 6.

More effort is required on the left-hand side. Here the object are marks on paper, the sign or representamen is the infinite sum $1 + 2 + 3 + \ldots$ and the initial interpretant is that it is an infinite process.

We must now take a brief pause to understand the mathematical explanation unfolding in this expression in order to be able to proceed with the semiosis. Though not complex, readers not interested in the details may skip a few paragraphs an see the final reasoning outlined on page 17.

As we said, the infinite sum is a divergent series. Another way to see this divergence is to consider the hyperharmonic series,

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \dots$$

For the exponent $s = -1$ we get exactly the sum of all natural numbers:

$$\frac{1}{1^{-1}} + \frac{1}{2^{-1}} + \frac{1}{3^{-1}} + \dots = 1 + 2 + 3 + \dots$$

In other words: the sum is equivalent to the value of the hyperharmonic series for $s = -1$. However, this series converges only for $s > 1$; for $s = -1$ the sums goes to infinity.

Now we consider a generalization of the hyperharmonic series, where we allow s to be a complex number. This generalization takes the name of the *Riemann zeta function*, and it is defined just like the hyperharmonic series, but for complex values of s:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

In the complex plane, the convergence condition that we had for real values of s still holds: the series only converges when $\mathrm{Re}(s) > 1$ and thus the Riemann zeta function is only defined in that region of the complex plane.

We said that the sum $1+2+3+\dots$ is the hyperharmonic series for $s = -1$, which is to say that it is the Riemann zeta function at -1: $\zeta(-1)$. But the ζ function is not defined there, right? Fortunately, since ζ is now a function in the complex plane, we can use a technique known as *analytic continuation*, which can extend the function beyond its original domain[7]. We do not care about the details of how this is done or even about what continuation we get (the formulas do not matter). We are only interested in that now we can get some value for the Riemann zeta function at every point of the complex plane:

$$\zeta(s) = \begin{cases} \sum_{n=1}^{\infty} \frac{1}{n^s} & \text{if } \mathrm{Re}(s) > 1 \\ (1 - 2^{1-s})^{-1} \cdot \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s} & \text{if } 0 < \mathrm{Re}(s) < 1 \\ 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \cdot \Gamma(1-s) \cdot \zeta(1-s) & \text{if } \mathrm{Re}(s) < 0 \end{cases}$$

In other words, $\zeta(-1)$ is now defined. Thus, we have:

$$1+2+3+\dots = \frac{1}{1^{-1}} + \frac{1}{2^{-1}} + \frac{1}{3^{-1}} + \dots =^* \zeta(-1) = 2^{-1}\pi^{-2} \sin\left(\frac{-\pi}{2}\right)\Gamma(2)\zeta(2) = -\frac{1}{12}$$

[7] An analogy might help understand what analytic continuation does. Imagine a picture that has been cut in half, and we only have the left-hand side. One might try to draw the right-hand side in such a way that completes the picture, trying to fit both sides. The same happens here. We have a function defined on a restricted domain, and we extend it to a bigger one in such a way that some of the original properties of the function are preserved.

The star $*$ shows the point where the chain of deductions is unrigorous: for $s = -1$ the zeta function is not defined as that infinite sum so that equality does not hold. Yet since it has been obtained through analytic continuation, these two values are somewhat related. In other words, when reading the sign $1 + 2 + 3 + \cdots = -\frac{1}{12}$ we should interpret the equality as a relation, a symbol connecting these two mathematical entities not in a strict identity, but in a more relaxed sense of an equivalence relation[8]: the equivalence relation of being "connected" by the analytic continuation of the Riemann zeta function.

Going back to the semiosis, this mathematical reasoning could be represented in the following four steps:

1. Initial premise: $1 + 2 + 3 + \ldots$ is an infinite process.
2. The infinite sum $1 + 2 + 3 + \ldots$ can be rewritten as:
$$\frac{1}{1^{-1}} + \frac{1}{2^{-1}} + \frac{1}{3^{-1}} + \ldots$$
3. The equivalent form corresponds to the value of the Riemman zeta function at -1:
$$\frac{1}{1^{-1}} + \frac{1}{2^{-1}} + \frac{1}{3^{-1}} + \cdots = \zeta(-1)$$
4. The value $\zeta(-1)$ can be computed thanks to analytic continuation, getting $-\frac{1}{12}$.

These four steps in the reasoning correspond to four chained semiosis. The complete process can be seen in Figure 7.

After four steps in the semiosis, we reach an interpretant that is exactly that of the semiosis launched for the right-hand side of the original sign: *finite object*. As a result, we can see that the two symbols agree and we can argue that we reached a final interpretant for the complete sign: the equality represents a prediction about signs that can be expressed as "the two sides are related by the equivalence relation of analytic continuation".

4 Conclusion

Throughout this paper I have shown a brief yet significant variety of semiotic models for a better understanding of mathematics and the philosophical issues related to it. Unfortunately, due to space limitations, I have just scratched the surface.

[8]A *relation* between two sets is called of *equivalence* if it is reflexive (every element is related to itself), symmetric (if a is related to b then b is related to a), and transitive (if a is related to b and b is related to c then a is related to c). The identity relation on natural numbers is an equivalence relation.

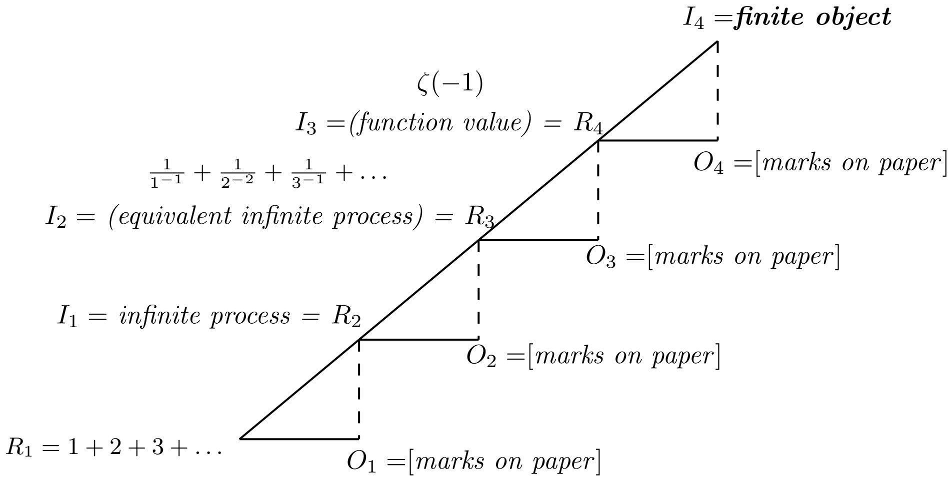

Figure 7: Semiosis for the left-hand side of the sign, conveying the mathematical reasoning outlined before.

Regarding section 2, where we discussed Rotman's corporeal semiotics for mathematics, I would have liked to explore some objections to his model, which are many and varied. For interested readers, a good critical review is that of Mortensen and Roberts (1997), who have also made their own contributions to semiotics and mathematics. Nevertheless, it is very clear that Rotman's semiotics offers a somewhat accurate and interesting framework for mathematical philosophy and, rather fairly, it is now considered one of the most significant contributions to postmodern epistemology and philosophy of science.

With regard to section 3, I am convinced that many other semiotic theories may be applicable to the understanding of mathematical signs. The ones applied were chosen due to their generality and also because they can work with a rather uncommon sign, one leading to a seeming contradiction. Besides, they are compatible with Rotman's analysis of mathematical language. Had we approached mathematics from a perspective based not so much on particular signs but on general mathematical language in culture, Yuri Lotman's cultural semiotics would have been appropriate and, in fact, I am sure there are fascinating analyses to be carried out with that perspective, which, unfortunately, had to be left out this essay.

All in all, it is clear that semiotics can offer new insights into the understanding of mathematical language and philosophy, and its confluence is a fascinating research field that is being and will continue to be further explored.

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